Q 6, 7, 8 will be replaced with Turing machines & countability

2016 Annual

**1b)** xQy ⇔ x-y = x2 - y2

⇔ x - y = (x - y)(x + y)

⇔ (x - y) - (x - y)(x + y) = 0

⇔ (x - y) (1 - (x + y)) = 0

⇔ x = y or 1 = x + y

**iii)** for all x,y e Z xQy and yQx => x = y

You cannot conclude that if a relation is symmetric it cannot be antisymmetric since there exists “=” that is both symmetric and antisymmetric.

**Disprove**: Antisymmetry directly

Construct a counterexample such that x + y = 1 holds and thus x != y. choose x=0 y=1 then

xQy and yQx, but x != y

no Q is not antisymmetric

iv) transitivity

for all x,y,z e Z, assume xQy and yQz.

Transitivity holds if xQz

consider 2 cases corresponding to xQy and 2 cases x=y or x+y=1

corresponding to yQ, hence 2x2=4 cases.

Case 1:

x = y and y = z => x = z by the transitivity of equality => xQz holds

Case 2:

x = y and y + z = 1 then x + z = 1 => xQz holds

Case 3:

x + y = 1 and y = z, then x + z = 1 => xQz holds

Case 4:

x + y = 1 and y + z = 1

y = 1 - x and y = 1- z => 1 - x = 1 - z

⇔ x - z = 1-1 ⇔ x - z = 0 ⇔ x = z => xQz holds

xQz is verified in all 4 cases => the relation Q is transitive

v) Determine whether Q is an equivalence relation

Q is reflexive, symmetric and transitive => it is an equivalence relation

vi) Determine whether Q is a partial order

For Q to be a partial order, Q would have to be reflexive, antisymmetric and transitive, but it fails to be antisymmetric => Q is not a partial order.

Q2 a) Let A be a semigroup. Show that if A has an identity element, then that identity element is unique.

Proven in class. Assume there exists e and e’ in A, both identity elements => for all a e A

a\*e = e\*a = a

a\*e’ = e’\*a = a

Consider e \* e’ = e’ because e is an identity element but e \* e’ = e because e’ is an identity element. So that means e = e’ so the identity element is unique.

b) Let A = {1,4,7} Consider the power set P(A) of A consisting of all subsets of A

Let P(A) be endowed with the set intersection ∩ operation

i) is (P(A), ∩) a semigroup? - binary operation & associative

Check ∩ binary operation & associative

set intersection is a binary operation because for any B,C in P(A) (B,C subsets of A)

their intersection is also a subset of A because B being a subset of A and C being a subset of A implies that their common elements B ∩ C is an element of A implies B ∩ C is an element of P(A).

Show set intersection is associative.

For any B,C subsets of P(A) recall that B ∩ C = { x is in A | x is in B and x is in C}

= {x is in A | x is in B and x is in C}

Associativity holds if for all b,c,d is an element of P(A) (B∩C)∩D = B∩(C∩D)

The logical connective “and” ∩ associative => (B∩C)∩D = B∩(C∩D) holds => ∩ is associative

ii) Is (P(A), ∩) a monoid? - semigroup, identity element

The identity element E is an element of P(A) has to satisfy that for all B e P(A), B∩E = E∩B = B

(P(A), ∩) is a monoid with identity element A

iii) is P(A), ∩ a group? For (P(A), ∩) to a group it has to be a monoid such that every element P(A) is invertible ie for all B e P(A) there exists B e P(A) such that B∩B-1 = B-1∩B = A - the identity

False. Some elements are invertible but not all.

We give a counterexample

let B = {1}, there does not exist B-1 e P(A) s + B∩B-1 = A because B∩B-1 subset of B

set B subset A, B != A

**Q3**

Describe the formal language over the alphabet {a, l, p} generated by the context-free grammar when only non-terminal is <S> where start symbol <S> and where production rules are the follow

1. <S> => l
2. <S> => a<S>
3. <S> => a<S>p

L = {am l pn | m,n e N, m >= n >= 0}

b) L is the language over the alphabet {0, 1} consisting of all words where the string 11 is a substring

L = { u11v | u,v e {0, 1} \*}

build a finite state acceptor for L



c) Device a reg grammar in normal form that generates L

Associate to i -> non terminal <S>

A -> non terminal <A>

B -> non terminal <B>

1. <S> => 0<S>
2. <S> => 1<A>
3. <A> => 0<S>
4. <A> => 1<B>
5. <B> => E

Turing machines

1. design a turing machine taht recognises L = {(10)n | n e N} = {e, 10, 1010, …}

The tape head is positioned over teh first cell

1. If \_ is in the cell, then accept   
   If 0 is in the cell reject
2. If 1 is in the cell, change 1 to x. move right
3. if \_ or 1 is in the cell then reject
4. Change 0 to y and move right
5. Go to step 1

Process 101

101

x01

xy1

xyx reject

7b transition diagram for ™ from a)